Jamming transitions and avalanches in the game of Dots-and-Boxes

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We study the game of Dots-and-Boxes from a statistical point of view. The early game can be treated as a case of random sequential adsorption, with a jamming transition that marks the beginning of the end game. We derive a set of differential equations to make predictions about the state of the lattice at the transition, and thus about the distribution of avalanches in the end game.

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Real-life games have traditionally inspired research by mathematicians, economists, psychologists, computer scientists, and also physicists, starting with the development of probability theory to handle games of chance [1–3]. Later fields of research were economic game theory [4,5], where usually two or more players have to make simultaneous and independent decisions without knowing what the other player is going to do, and combinatorial game theory [6,7], where players take alternating turns, and all information on the state of the game and the possible future moves is available.

The popular children's game of Dots-and-Boxes falls into the latter category. The game is played on a rectangular lattice (a checkered sheet of paper), and players take alternating turns. At each turn, the active player occupies an edge. If he thus occupies the fourth edge of at least one of the two adjacent squares, the player continues the turn by placing another edge. Rules vary on whether a player must take a square if he has the chance—the option not to take it allows for a number of subtle moves (for a guide to the end game of Dots-and-Boxes, see Ref. [6], Vol. 2); however, for simplicity, we demand that any chance to take a square must be used. The game ends when all edges and squares are occupied. The player who took more squares wins.

The game can be separated into two distinct phases: in the early game, players usually occupy edges more or less at random. However, they avoid placing the third edge around any square, which would give the opponent the opportunity to score. Phase 1 ends when this is no longer possible: all free edges have at least one adjacent square with two occupied edges. This situation, which is shown in Fig. 1, is analogous to a jamming transition in models of random sequential adsorption (RSA). The main focus of this paper is the modeling of the early game, using methods from RSA theory and Monte Carlo simulations, and to determine the time and state of the game at the jamming transition.

In the end game, squares are rapidly filled: each edge creates an opportunity to score, which often triggers another opportunity, and another, until the avalanche is terminated somehow. The end game is largely determined by nonlocal strategies (like figuring out what the shortest possible avalanches are), and thus not accessible to the methods used for the early game. However, the state of the game at the beginning of the end game limits the options available for the rest of the game, and allows for predictions of the minimum number of turns until the end of the game, the average size of avalanches, and so forth.

I. THE MODEL

As a model for the game that can be both simulated with reasonable computational effort and handled analytically, we use the following rules of behavior for the players:

(1) *Occupy squares if possible*. In accordance with the rules that prescribe a greedy strategy, the active player searches the board for all free edges with an adjacent square with exactly three occupied edges around it, picks one of them at random, occupies it, and continues his turn. This is repeated as often as possible.

(2) *Create no opportunities for the opponent*. If there are

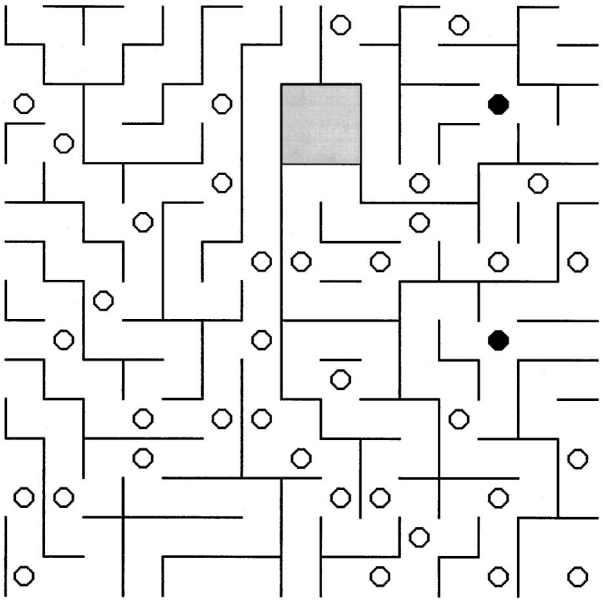

FIG. 1. A typical state of the game at the transition from the early game to the end game. Empty circles mark single defects (explained in Sec. III), closed circles denote double defects, and shading denotes a closed area.

no squares to be taken, the player looks for free edges whose adjacent squares both have less than two occupied edges. One of those is picked at random and occupied, and the turn ends.

(3) *Minimize the opponent's score*. If both prescriptions (1) and (2) yield no suitable edge, the active player has to pick the third edge around some square, thus giving his opponent a chance to take it according to (1). He checks all edges to see how many points his opponent would gain from them. He picks one of those that give away the smallest number of points, occupies it, and ends the turn.

To get rid of boundary effects, the lattice on which the game is played is assumed to have periodic boundary conditions. Furthermore, the analytical treatment is strictly valid in the limit of infinite lattice size only.

The lattice has N squares. Accordingly, the number of edges is $2N$. The number of turns is counted by T. The rescaled time $t=T/2N$ runs from 0 to at most 1; however, since more than one edge per turn is occupied in the end game, the game ends at times $t<1$. Two other useful quantities to describe the system are the number P of occupied edges (or $p=P/2N$, the probability that a given edge is occupied), and the number of filled squares S (or $s=S/N$, respectively).

II. THE EARLY GAME

As mentioned in the Introduction, the early game can be treated as a special case of RSA [8]. The basic idea of RSA is to deposit particles (atoms, dimers or, in our case, edges) on randomly chosen sites of a surface (often on a regular lattice) unless this deposition violates restrictions posed by particles that were adsorbed before. In Dots-and-Boxes, the edges form a square lattice with a peculiar short-range three-particle repulsion.

The usual procedure to treat RSA problems analytically is to solve a set of coupled differential equations (ODEs) that describe the probability of encountering the various possible configurations of particles (for details, see Ref. [8]). Unfortunately, this set of ODEs is generally not closed, i.e., the equations for small configurations include probabilities of larger configuration, which in turn depend on still larger configurations. At some point, one has to neglect correlations and truncate the equations. Since the number of terms increases dramatically with increasing order of truncation, we will use the simplest approximation that still captures the interaction correctly.

This approximation characterizes each free edge by two indices—the first index i for the number of occupied edges surrounding the adjacent square above or to the left of the considered edge, the second index j for the occupation number of the square below or to the right (for example, Fig. 2 shows all configurations with index 21 around a horizontal edge). One can then count the number F_{ij} of free edges with indices ij, and determine their density $f_{ij}=F_{ij}/2N$.

All possible configurations with the same indices are now assumed to have the same probability, and correlations beyond nearest neighbors are neglected. For example, if we consider a free edge with index 11, we have no information

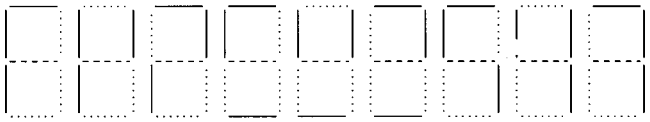

FIG. 2. Free edges are classified according to the number of occupied edges in adjacent squares. In all cases shown here, the central edge (dashed line) has the index 21. All possible configurations with the same index ij are assumed to have equal probabilities.

on the free edges surrounding its adjacent squares, except that one of their indices must be 1. Accordingly, we assume that the other index (let us say, j) follows a simple conditional probability, $\mathrm{Prob}(j|i)=f_{ij}/\Sigma_k f_{ik}$.

In the inititial phase of the game, there is an extensive number of edges that can be occupied without giving the opponent an opportunity to score, following prescription (2) of the strategy. These are edges from the categories 00, 10, 01, and 11. Now we can see what happens to the f_{ij} if an edge is occupied, i.e., a time step $dt=1/2N$ elapses. The chosen edge (let us say, it has indices kl) is occupied, and f_{kl} is decreased by $df_{kl}=1/2N$. This means that for all edges that can be chosen, the differential equation for f_{kl} includes a loss term equal to the probability that an edge with index kl is picked:

$$\frac{df_{kl}}{dt}=-\frac{f_{kl}}{f_f}+\cdots \qquad \text{for} \quad k,l\in\{0,1\}, \tag{1}$$

where f_f is the density of edges that can be occupied before the transition,

$$f_f=f_{00}+f_{01}+f_{10}+f_{11}. \tag{2}$$

The free edges in the squares next to the chosen edge must be updated: the index corresponding to the considered square is increased by one. For instance, assume that the upper/left adjacent square of the chosen edge has occupation number 0 [which happens with probability $(f_{00}+f_{01})/f_f$]. Two of the three remaining edges around that square will then have their second index increased by one, whereas one will have its first index incremented. This leads to gain and loss terms in the ODEs:

$$f_{i0}:-2\frac{f_{i0}}{\sum_{k=0}^{2} f_{k0}}\frac{f_{00}+f_{01}}{f_f},$$

$$f_{0i}:-\frac{f_{0i}}{\sum_{k=0}^{2} f_{0k}}\frac{f_{00}+f_{01}}{f_f},$$

$$f_{i1}:+2\frac{f_{i0}}{\sum_{k=0}^{2} f_{k0}}\frac{f_{00}+f_{01}}{f_f},$$

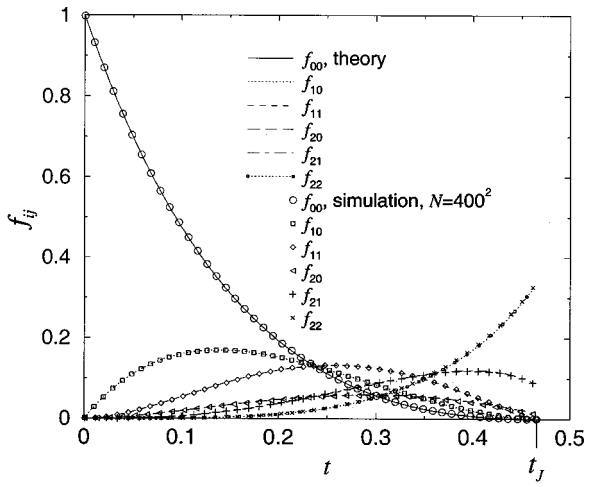

FIG. 3. Comparison between the numerical solution of the ODEs and a simulation with $N=400\times400$.

$$f_{1i} := \frac{f_{0i}}{\sum_{k=0}^{2} f_{0k}} \frac{f_{00}+f_{01}}{f_f}. \tag{3}$$

What follows is a rather tedious summation of terms for the possible combinations of indices.

The equations derived in this fashion can be simplified considerably by assuming the symmetry $f_{ij}=f_{ji}$, thus keeping only the categories with $i \geqslant j$. With the abbreviations

$$r_0=(f_{00}+f_{10})/(f_{00}+f_{10}+f_{20}), \tag{4}$$

$$r_1=(f_{10}+f_{11})/(f_{10}+f_{11}+f_{21}), \tag{5}$$

the resulting system of differential equations looks as follows:

$$\frac{df_{00}}{dt}=(-f_{00}-6f_{00}r_0)/f_f,$$

$$\frac{df_{10}}{dt}=(-f_{10}+3f_{00}r_0-3f_{10}r_0-2f_{10}r_1)/f_f,$$

$$\frac{df_{11}}{dt}=(-f_{11}+6f_{10}r_0-4f_{11}r_1)/f_f,$$

$$\frac{df_{20}}{dt}=(-3f_{20}r_0+2f_{10}r_1)/f_f,$$

$$\frac{df_{21}}{dt}=(-2f_{21}r_1+2f_{11}r_1+3f_{20}r_0)/f_f,$$

$$\frac{df_{22}}{dt}=4f_{21}r_1/f_f. \tag{6}$$

This can be solved numerically and compared to simulations, as seen in Fig. 3. The agreement is very good, but becomes slightly worse close to the jamming transition—the point where f_f becomes zero and the early game ends. The accuracy could probably be improved by including probabilities of larger configurations; however, it is not worth the effort, since the calculated numbers agree with results from real play only in the order of magnitude anyway (see Sec. V). The predictions and numerical values for the jamming time t_J and the order parameters at that time are given in Table I.

TABLE I. Order parameters at the jamming transition for the square lattice.

	Theory	Simulation
t_J	0.4615	0.4657
f_{22}	0.3244	0.3409
f_{21}	0.0901	0.0846
f_{20}	0.0169	0.0121

III. THE END GAME

When the last edge from the categories 00, 01, 10, and 11 has been taken, prescription (2) is no longer an option. The game now alternates between prescriptions (1) and (3). Each player's turn begins by filling the squares of the avalanche that his opponent has offered him by placing the third edge around some square [prescription (1)]. When all possible squares have been taken, the active player now determines the avalanche that his opponent must take [prescription (3)]. Since the length of the avalanche triggered by placing an edge is not a local property of that edge (and highly correlated to that of neighboring edges), a description by differential equations analogous to the early game makes little sense. However, since the state of the system at the transition largely determines the options of the players later on, we can make quantitative predictions about the end game from the knowledge gained in Sec. II.

Figure 1 shows the state of a game with 15×15 squares at the jamming transition. The squares are segments of a tunnel if they have two occupied edges. They represent tunnel branchings (or *single defects*, marked by empty circles) if they have only one edge, and tunnel crossings (or *double defects*, marked by full circles) if they have none.

Thus, three edges from the f_{21} category form a single defect, whereas four f_{20} edges make up a double defect. The density of defects can be calculated directly from the order parameters at the transition.

As an additional check, one can make sure that the total number of edges (both occupied and free) add up to $2N$, and that the number of squares (tunnels and defects) add up to N. This leads to the following equations:

$$t_J+2f_{20}+2f_{21}+f_{22}=1, \tag{7}$$

$$3f_{20}+(10/3)f_{21}+2f_{22}=1, \tag{8}$$

which are fulfilled for both the analytical and the experimental values.

This enables us to make some statements about the avalanches or chains of occupied squares that occur in the late

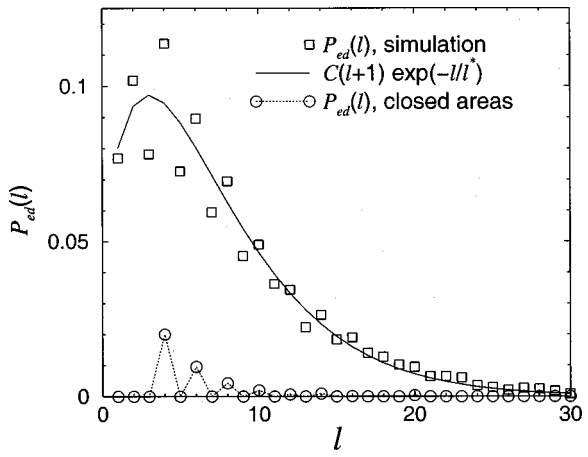

FIG. 4. Probablility distribution $P_{ed}(l)$ of avalanche lengths l at the jamming transition, averaged over randomly chosen free edges. Simulations average over 40 runs with $N=50\times 50$.

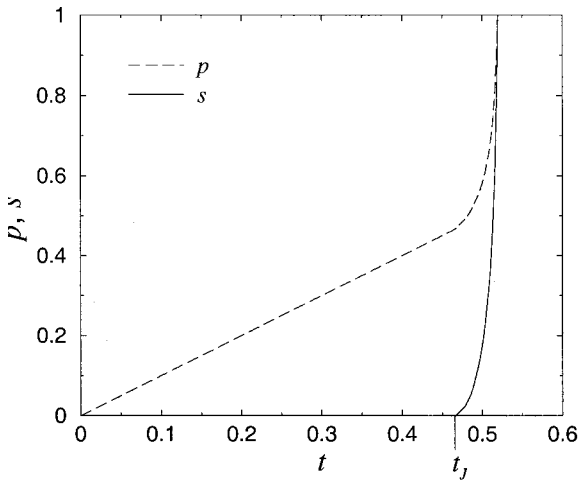

FIG. 5. Time development of fraction of occupied edges (p) and squares (s) in a simulation on an 80×80 lattice.

game. An avalanche started in a tunnel fills the tunnel and ends at the defect edges on both sides of the tunnel. Thus, with $4N(f_{21}+f_{20})$ defect edges, there are at least $N_A=2N(f_{21}+f_{20})$ different potential avalanches at the time of the transition. (For now, we neglect avalanches in closed areas and other complications.) We can also calculate the number of tunnel segments (squares with occupation 2) from the order parameters: $N_T=N-(4/3)Nf_{21}-Nf_{20}$. This yields an average length of the tunnel segments of $N_T/N_A\approx 4.5$ for the values of f_{ij} from the simulation, as given above.

Note that in an analogy to the ''waiting time paradox,'' the average avalanche length becomes larger than the mentioned value of 4.5 if avalanches are started at randomly chosen edges rather than randomly chosen tunnel segments, because longer tunnels include more edges and are thus chosen with higher probability

Let us assume that the probability distribution $P_{av}(l)$ of tunnel lengths l follows an exponential with a decay constant $1/l^*$. Since $l\geqslant 1$, the normalization constant is $\exp(1/l^*)-1$, and the average value is

$$\langle l\rangle_{av}=\sum_{l=1}^{\infty} l(e^{1/l^*}-1)e^{-l/l^*}=\frac{1}{1-e^{-1/l^*}}. \tag{9}$$

With the mentioned value of $\langle l\rangle_{av}=4.5$, one gets $l^*\approx 4.0$ and $P_{av}(l)\approx 0.284\exp(-l/4.00)$. Since each tunnel avalanche of l squares length has $l+1$ edges where it can be started, the probability distribution of avalanche lengths averaged over free edges follows the form $P_{ed}(l)\propto(l+1)\exp(-l/l^*)$.

This agrees fairly well with simulations, as seen in Fig. 4. However, there is a preference for even avalanche lengths, which can partly be explained with the presence of *closed areas*. These are areas that contain no defects and are separated from the rest of the board by occupied edges. They include an even number ($\geqslant 4$) of squares. The probability of an edge being in a closed area of size l is shown in Fig. 4 (open circles). Even if it is taken into account, even avalanches are more likely, for reasons that are still unclear.

The density of defects at the transition allows us for a prediction of the total number of turns in the end game. When an avalanche is terminated at a defect, that defect is turned into a tunnel segment (if it was a single defect before) or into a single defect (if it was a double defect). Therefore the number of defect-terminated avalanches N_{DTA} is half the number of single defects plus the number of double defects:

$$N_{DTA}=[(2/3)f_{21}+f_{20}]/N\approx 0.069N. \tag{10}$$

This means that the time difference from t_J to the end of the game t_E should be at least

$$t_E-t_J\geqslant N_{DTA}/2N=0.034. \tag{11}$$

However, this is really only a lower bound on the time found in simulations, $t_E-t_J\approx 0.054$. The reason for the deviation is the existence of avalanches that do not change the number of defects, namely, avalanches in closed areas. These areas are not necessarily present at the beginning of the end game. Instead, they may initially be half-closed areas: areas that are not quite closed, but connected to the rest of the system by a single defect. Depending on whether the avalanche is started in the tunnel outside the half-closed area or inside it, it is either turned into a closed area, or it is filled, the defect is removed, and the adjacent tunnel is filled as well. Since it is usually desirable to give the opponent as few points as possible, most of the half-closed areas in real play will be turned into closed areas, and then filled.

Apart from these exceptions, avalanches tend to get dramatically longer as the end game goes on (see Fig. 5). This is due to two effects: First, small avalanches are triggered earlier than larger ones due to prescription (3), and thus removed; second, avalanches that stop at a single defect turn it into a tunnel segment, merging two potential avalanches into one.

IV. OTHER LATTICES

The game can be played on lattices other than the square, as long as there is a notion of an edge separating two cells,

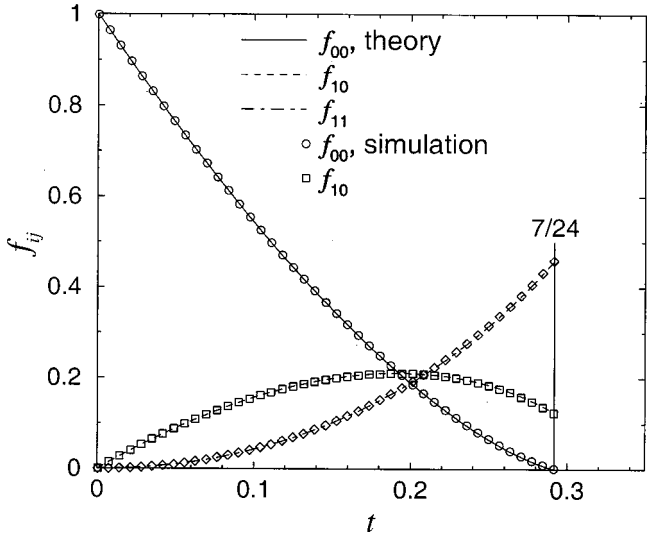

FIG. 6. Comparison between the analytical solution [Eqs. (15)–(18)] of the ODEs and a simulation on a triangular lattice with $N = 20\,000$.

and placing a single edge is not enough to make scoring possible. The simplest case is the triangular lattice, where there are only single defects, and the relevant order parameters in the early game are f_{00}, f_{10}, and f_{11}. The corresponding differential equations are

$$\frac{df_{00}}{dt} = -1 - 4\frac{f_{00}}{f_{00}+f_{10}}, \tag{12}$$

$$\frac{df_{10}}{dt} = 2\frac{f_{00}-f_{10}}{f_{00}+f_{10}}, \tag{13}$$

$$\frac{df_{11}}{dt} = 4\frac{f_{10}}{f_{00}+f_{10}}. \tag{14}$$

They can even be solved analytically by introducing a rescaled time τ with $d\tau = (f_{00}+f_{10})^{-1}dt$, and solving the remaining system of linear ODEs in τ with constant coefficients. One gets

$$f_{00}(\tau) = e^{-4\tau}(2-e^{\tau}), \tag{15}$$

$$f_{10}(\tau) = 2e^{-4\tau}(e^{\tau}-1), \tag{16}$$

$$f_{11}(\tau) = 2e^{-4\tau} - (8/3)e^{-3\tau} + 2/3. \tag{17}$$

Using Eqs. (15) and (16), t can be calculated:

$$t(\tau) = (1-e^{-3\tau})/3; \quad \tau(t) = -(1/3)\ln(1-3t). \tag{18}$$

Again, agreement between theory and simulation is very good, as seen in Fig. 6. The predicted and observed values for the jamming transition and the order parameters are given in Table II.

Other possible lattices include hexagonal and three-dimensional (3D) cubic lattices. In the latter case, edges correspond to faces of unit cubes. Interestingly, the differential equations are the same for both the hexagonal and the 3D cubic lattice, since both have six edges/faces surrounding each hexagon/cube. Although structurally simple, the equations involve 15 order parameters and are not written out for the sake of brevity.

TABLE II. Order parameters at the transition for the triangular lattice.

	Theory	Simulation
t_J	7/24=0.291 667	0.2930±0.0002
f_{10}	1/8=0.125	0.1209±0.0003
f_{11}	11/24=0.458 33	0.4655±0.0002

Of course, the whole range from single to quadruple defects can occur; however, multiple defects are rarer than single ones, as seen in Table III. Results from simulations confirm the picture predicted by calculations, with the usual deviations of the order of 10^{-3}.

Is the game still interesting on other lattices? Disregarding the practical difficulties of playing on a 3D cubic lattice, all basic mechanisms of the game still work, including closed and half-closed areas. A rough estimate shows that the initial avalanche length (averaged over possible avalanches) is 4.8 for the triangular lattice and 3.6 for the 3D cubic and hexagonal lattice, similar to the square lattice. We therefore expect that real-life games on other lattices would not be much different from Dots-and-Boxes on regular square lattices.

V. COMPARISON TO REAL PLAY

We let some co-workers play a computer version of Dots-and-Boxes (with periodic boundary conditions and $N = 10 \times 10$) to see if their style of play is well described by the assumptions in Sec. I. Generally speaking, human players did not place edges at random in the early game; instead they tended to add edges to existing structures. In some cases, this led to a significantly lower number of defects, and thus longer avalanches. One pair of players chose to get rid of the periodic boundary conditions by drawing a frame around the board early in the game.

Nevertheless, some games showed quantitative similarities to our theoretical predictions. The order parameters from one of these games are shown in Figs. 7 and 8.

Our test players did not try tactical subtleties, such as adjusting the number of avalanches in order to get the last (and presumably longest) one. They were usually happy if

TABLE III. Order parameters for the 3D cubic and hexagonal lattice.

	Theory	Simulation 3D	Simulation hexagonal
t_J	0.6367	0.6381±0.0001	0.6351±0.0001
f_{40}	0.000 17	0.000 10±0.000 03	0.000 30±0.000 03
f_{41}	0.0027	0.0022±0.00001	0.0036±0.0001
f_{42}	0.0177	0.0162±0.0002	0.0198±0.0002
f_{43}	0.0577	0.057 23±0.0002	0.0579±0.0002
f_{44}	0.2064	0.2105±0.0002	0.2018±0.0002

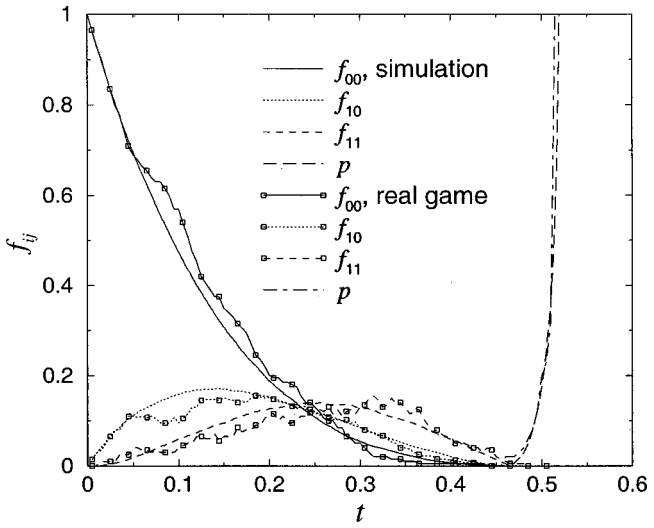

FIG. 7. Comparison between a simulation with $N=80\times80$ and a real-life game on a 10×10 lattice. To avoid confusion, only the curves for f_{00}, f_{10}, f_{11}, and the density of filled squares p is shown in this figure (see also Fig. 8).

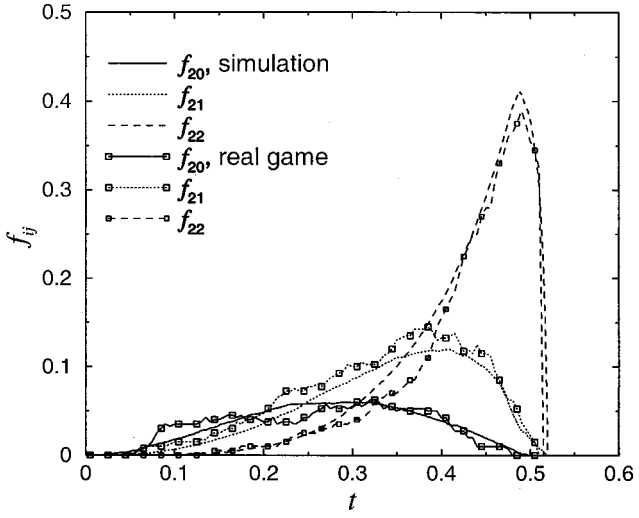

FIG. 8. The remaining order parameters of the game shown in Fig. 7.

they avoided blatant mistakes.

Of course, one could include human tendencies and extend the model to include *cooperative sequential adsorption* [8], where edges are preferably placed next to edges occupied before. However, since this could not describe all human players with the same set of parameters and would probably give no qualitative new insights, the usefulness of this extension is questionable.

VI. SUMMARY AND CONCLUSION

We gave a statistical treatment of the game of Dots-and-Boxes, using some simplifying assumptions for the behavior of the players. In the early game, since a finite fraction of edges can be chosen, a mean-field description given by a system of coupled differential equations works well. It makes predictions about the point where avalanches start and the degree of geometrical frustration at that point. The same scheme works for all kinds of regular lattices; the relevant quantity is the number of edges or faces around a cell, such that hexagonal and 3D cubic lattices are described by the same equations.

These predictions allow us to make statements regarding the statistics of avalanches, as well as the total number of turns in the end game. The presence of closed and half-closed areas makes the situation more complicated; unfortunately, they cannot be captured by the approximations made in the calculation of the early game.

While results from calculations give good agreement with simulations, human players have various habits that cannot be easily included in an all-encompassing mean-field treatment (''I like making corners. They look nice.''). Thus, while our analysis has yielded some insight in the underlying processes of Dots-and-Boxes, quantitative agreement with human play is not always satisfactory.

ACKNOWLEDGMENTS

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